# Adaptive Generation of Rational Function Approximations for Microwave Network Parameters

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***Abstract*— This paper introduces a new method for rational function approximation of measured or simulated microwave network parameters. The proposed adaptive generation (AG) approach allows estimation of the model order for a given error tolerance and does not require any initial estimates or adjustment of hyperparameters. We present an implementation of AG using the recently developed orthogonal rational approximation (ORA) method.**

***Keywords*— rational functions, scattering parameters, stability, macromodeling, vector fitting.**

## I. Introduction

A common approach that is used to design and characterize microwave components is to perform frequency-domain electromagnetic simulations and measurements. Integration of these microwave network parameters into a circuit simulator can be achieved by approximating the frequency-domain data with a rational function. Methods available to generate rational function approximations from tabulated simulated or measured data include the widely popular vector fitting [1], Loewner framework [2], Sanathanan-Koerner iteration [3], RKFIT [4], ORA [5], and AAA (adaptive Antoulas-Anderson) [6], [7], [8] (see, e.g., [9] for a survey of additional methods).

The main theme of this paper is fitting a rational function of the Laplace variable $s$

$$R(s) = \frac{F(s)}{a(s)} = \frac{\sum_{i=0}^{n} F_i s^i}{\sum_{i=0}^{n} a_i s^i} \tag{1}$$

to measured or simulated data $H_i$ at $l$ frequency points $s_i$. The transfer function in (1) represents a blackbox macromodel that characterizes the component, where the roots of the numerator and denominator are denoted as zeros and poles respectively [10]. The objective is to solve the nonlinear least squares problem

$$\text{minimize} \quad \sum_{i=1}^{l} ||R(s_i) - H_i||_F^2 \tag{2}$$

where we use the Frobenius norm. The data $H_i$ and the numerator coefficients $F_i$ are matrices in general, whereas a scalar denominator polynomial $a(s)$ will be assumed. In the following, the compact expression $F(s)/a(s) \approx H$ will be used to denote the least squares problem. Typically, the linearized version $F(s) \approx a(s)H$ is solved iteratively in a heuristic approach.

The modeling methods outlined earlier continue to be used and developed to analyze a variety of microwave systems. For example, in [11], the scattering parameters of a printed circuit board with 38 embedded vias were obtained using the Loewner matrix framework with a new stability enforcement approach. In [12], the calculated S-parameters of 64 coupled transmission lines, and the simulated S-parameters of five 22-port stripline structures with multiple layers, were fit with a passivity-enforced Loewner matrix model. These macromodeling techniques may also be used for applications like parametric modeling in [13], where vector fitting was used in the design of microwave filters.

The rational function approximation problem has a long and broad history, so clarifying the significance and novelty of this approach is a pressing issue at this point. The paper will outline a new adaptive generation (AG) approach that is perhaps surprising in its simplicity to linearize the least-squares problem of (2). AG also provides an estimation of the model order $n$ – an often overlooked but nevertheless important problem. The monomial basis in (1) is, of course, ill-conditioned for large orders. AG is applicable to the well-trusted partial-fraction or barycentric bases, as well as the recent data-centered orthogonal rational basis [5], which will be the method used in this paper.

## II. Current Approaches for Rational Function Approximation

There are two major approaches for rational function approximation:

1) Sanathanan-Koerner (SK) iteration: Various implementations include RKFIT, stabilized SK, and the popular vector fitting algorithms. SK iteration is known to be accurate for approximating data with noise. The model order is fixed.
2) Adaptive Antoulas-Anderson (AAA): The barycentric basis of AAA comes from the Loewner framework. AAA is known to be accurate for closed-form functions or data without noise. The adaptive nature of AAA allows to estimate the model order.

The new adaptive generation (AG) approach in this paper can be considered as the third approach. They are all based on an iteratively updated polynomial $\hat{a}(s)$ as

$$R(s) = \frac{F(s)}{a(s)} = \frac{\frac{F(s)}{\hat{a}(s)}}{\frac{a(s)}{\hat{a}(s)}} = \frac{\frac{\sum_{i=0}^{n} F_i s^i}{\sum_{i=0}^{n} \hat{a}_i s^i}}{\frac{\sum_{i=0}^{n} a_i s^i}{\sum_{i=0}^{n} \hat{a}_i s^i}}, \tag{3}$$

where the linearized least squares problem becomes $F(s)/\hat{a}(s) \approx Ha(s)/\hat{a}(s)$. The choice of the iteration polynomial $\hat{a}(s)$ is critical in the convergence of the linearized

problem to the solution of the actual non-linear problem $F(s)/a(s) \approx H$. This choice is the main difference between the three approaches.

### A. Sanathanan-Koerner (SK) Iteration

In SK iteration, typical initial choices for the iteration polynomial are $\hat{a}(s) = 1$ or an estimate with slightly damped complex roots. The monomial basis in (3) is of course not suitable for large-order approximations and replaced with a better-conditioned basis such as a partial fraction (vector fitting) or an orthogonal rational basis (RKFIT or ORA [5]). For the example of vector fitting, the rational function can be expressed in barycentric form as

$$R(s) = \frac{\frac{F(s)}{\hat{a}(s)}}{\frac{a(s)}{\hat{a}(s)}} = \frac{K_0 + \sum_{i=1}^{n} \frac{K_i}{s-\hat{p}_i}}{k_0 + \sum_{i=1}^{n} \frac{k_i}{s-\hat{p}_i}} \tag{4}$$

where $\hat{p}_i$ are the roots of $\hat{a}(s)$. The solution of the linearized problem yields an updated denominator $a(s)$. In the next iteration, this updated denominator $a(s)$ is used in place of $\hat{a}(s)$ and a new least squares problem is solved. If $\hat{a}(s)$ approaches $a(s)$, the linearized problem $F(s)/\hat{a}(s) \approx Ha(s)/\hat{a}(s)$ converges to the solution of the actual nonlinear problem $F(s)/a(s) \approx H$. Note that $a(s)$ and $\hat{a}(s)$ have the same polynomial degree $n$. A disadvantage of SK iteration is that the model order is fixed (i.e., $n$ is assumed to be known.)

### B. Adaptive Antoulas-Anderson (AAA)

The AAA algorithm is based on choosing zeros of $\hat{a}(s)$ from a subset of the $l$ frequency points as $\hat{p}_i = \hat{s}_i$, resulting in the rational function in barycentric form

$$R(s) = \frac{\frac{F(s)}{\hat{a}(s)}}{\frac{a(s)}{\hat{a}(s)}} = \frac{\sum_{i=0}^{n} \frac{b_i \hat{H}_i}{s-\hat{p}_i}}{\sum_{i=0}^{n} \frac{b_i}{s-\hat{p}_i}}. \tag{5}$$

Evaluated at such a node $s = \hat{s}_i$, it can be seen that the barycentric form yields the interpolatory result $R(\hat{s}_i) = \hat{H}_i$ (where $\hat{H}_i$ is the data provided at frequency $\hat{s}_i$), independent of the weight $b_i$, as long as $b_i \neq 0$. The weights $b_i$ are calculated to fit to the remaining $l-(n+1)$ frequency points by solving the linearized least squares problem as in SK iteration. Then the frequency point with the maximum deviation is selected as the next node $\hat{s}_{n+1}$, and the iteration continues until a certain error tolerance is reached. This adaptive nature of AAA allows to estimate the model order, as each iteration increments the model order by one. This is an important feature, as a higher-than-necessary model order can result in spurious poles (poles with very small residues), affecting the model's accuracy.

## III. Adaptive Generation: Adaptive Order Estimation of AAA + Accuracy of SK Iteration

The denominator in the barycentric form of AAA (5) has no constant or linear terms, whereas SK in (3) has the linear term $k_0$. Consider now a slight modification of the barycentric form by adding a linear term $sk_\infty$ to the denominator:

$$R(s) = \frac{\frac{F(s)}{\hat{a}(s)}}{\frac{a(s)}{\hat{a}(s)}} = \frac{K_0 + sK_\infty + \sum_{i=1}^{n-1} \frac{K_i}{s-\hat{p}_i}}{k_0 + sk_\infty + \sum_{i=1}^{n-1} \frac{k_i}{s-\hat{p}_i}} \tag{6}$$

The rational function representation in the barycentric form of (6) is the basis for the proposed adaptive generation (AG) algorithm. Unlike SK iteration, the order $n$ is not assumed to be known and can now be estimated in an adaptive manner similar to AAA. However, the iterations are aimed at achieving $\hat{a}(s) \approx a(s)$, similar to SK. The main feature of AG is that the order of $a(s)$ is one more than the order of $\hat{a}(s)$, due to the addition of the linear term. Hence each iteration increments the model order $n$ by one. In contrast, SK iteration maintains the same model order over iterations. The proposed algorithm follows these main steps:

1) Start with $n = 1$ and $\hat{a}(s) = 1$.
2) Solve the linearized problem $F(s)/\hat{a}(s) \approx Ha(s)/\hat{a}(s)$ to calculate $a(s)$.
3) Solve the linear problem $F(s)/a(s) \approx H$ to calculate $F(s)$ using $a(s)$ from Step 2 and the residual error $\epsilon$.
4) If $n < n_{max}$ and $\epsilon > \epsilon_{max}$, set $n = n+1$, $\hat{a}(s) = a(s)$, and go to Step 2.

Note how Steps 2 and 3 are used to solve the denominator and numerator, respectively. Even though both of them can be obtained from the linearized problem in Step 2, separately solving the linear problem in Step 3 increases the accuracy, similar to the pole and residue extraction steps in vector fitting.

The accuracy of the linearized problem $F(s)/\hat{a}(s) \approx Ha(s)/\hat{a}(s)$ to solve the actual nonlinear problem $F(s)/a(s) \approx H$ depends on the convergence of $\hat{a}(s)$ to $a(s)$. In AG, increasing the model order in each step is meant to provide a better approximation of $\hat{a}(s)$ to $a(s)$. This heuristic goal is the same as in SK iteration. AAA, on the other hand, relies on the interpolatory nature of the barycentric form. Obviously convergence of $\hat{a}(s)$ to $a(s)$ is not expected in AAA, as the roots of $a(s)$ will in general not be a subset of the frequency points. On the other hand, each iteration increases the number of frequency points where interpolation occurs, with the goal of reaching a better least squares approximation overall. This interpolatory nature of AAA makes it suboptimal for least squares approximation of data including noise. The new AG approach therefore promises to incorporate the powerful features of both SK and AAA as shown in Table 1.

## IV. Numerical Results

Two examples were used to compare the performance of AG with vector fitting using the vectfit3 MATLAB package with lower-triangular parameter values. This was done in order to match the operations performed in vectfit3. The noisy scattering parameter data from a stripline, which was measured from 100 MHz to 110 GHz on a vector network analyzer, was modeled up to an order of 50, corresponding to 50 poles, as depicted in Figure 1. As shown in Figure 2, at model orders

Table 1. Comparison of Sanathanan-Koerner (SK), adaptive Antoulas-Anderson (AAA), and the proposed adaptive generation (AG) approaches. Superscript $(t)$ represents the values at the $t$th iteration.

| Method | Model Order | Noninterpolatory | Iteration Algorithm ($t \to t+1$) |
|---|---|---|---|
| SK | Fixed | Yes | $\frac{K_0^{(t)}+\sum_{i=1}^{n}\frac{K_i^{(t)}}{s-\hat{p}_i^{(t)}}}{k_0^{(t)}+\sum_{i=1}^{n}\frac{k_i^{(t)}}{s-\hat{p}_i^{(t)}}} \to \frac{K_0^{(t+1)}+\sum_{i=1}^{n}\frac{K_i^{(t+1)}}{s-\hat{p}_i^{(t+1)}}}{k_0^{(t+1)}+\sum_{i=1}^{n}\frac{k_i^{(t+1)}}{s-\hat{p}_i^{(t+1)}}}$ |
| AAA | Adaptive | No | $\frac{\sum_{i=0}^{t}\frac{b_i^{(t)}\hat{H}_i}{s-\hat{p}_i}}{\sum_{i=0}^{t}\frac{b_i^{(t)}}{s-\hat{p}_i}} \to \frac{\sum_{i=0}^{t+1}\frac{b_i^{(t+1)}\hat{H}_i}{s-\hat{p}_i}}{\sum_{i=0}^{t+1}\frac{b_i^{(t+1)}}{s-\hat{p}_i}}$ |
| AG | Adaptive | Yes | $\frac{K_0^{(t)}+sK_\infty^{(t)}+\sum_{i=1}^{t-1}\frac{K_i^{(t)}}{s-\hat{p}_i^{(t)}}}{k_0^{(t)}+sk_\infty^{(t)}+\sum_{i=1}^{t-1}\frac{k_i^{(t)}}{s-\hat{p}_i^{(t)}}} \to \frac{K_0^{(t+1)}+sK_\infty^{(t+1)}+\sum_{i=1}^{t}\frac{K_i^{(t+1)}}{s-\hat{p}_i^{(t+1)}}}{k_0^{(t+1)}+sk_\infty^{(t+1)}+\sum_{i=1}^{t}\frac{k_i^{(t+1)}}{s-\hat{p}_i^{(t+1)}}}$ |

higher than about 43, the approximation by AG had lower RMS error than that of vector fitting for the two-port model.

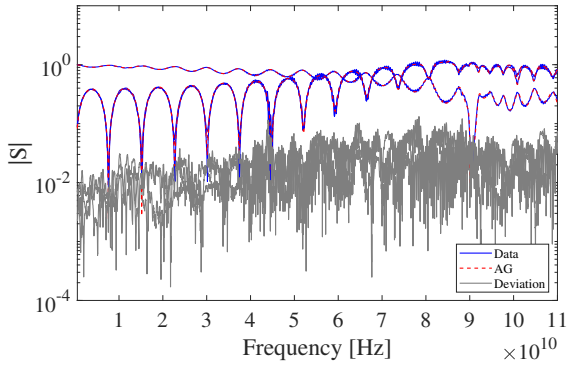


Fig. 1. Example 1: Noisy stripline data measured at 5001 frequency points was fitted using AG

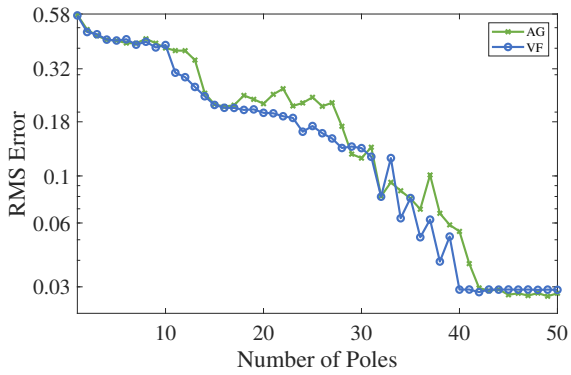


Fig. 2. Example 1: RMS error for AG was lower than that of vector fitting at higher orders

A second example, shown in Figure 3 and Figure 4 depicts the scattering parameters from a cavity resonator resulting from a full-wave simulation using Sonnet at ten arbitrarily-located ports. In a comparison of model orders up to 100, Figure 4 shows that models with orders above approximately 85 had lower RMS error when constructed with AG than with vector fitting.

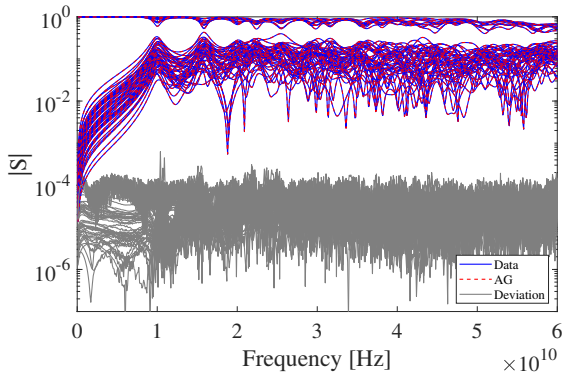


Fig. 3. Example 2: Simulated cavity resonator data at 10 ports with 600 frequency points was fitted using AG

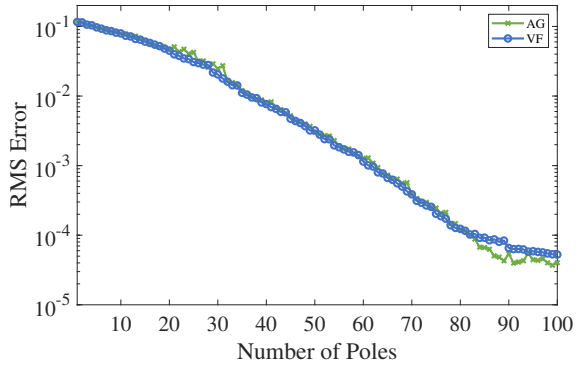


Fig. 4. Example 2: At higher orders, AG had a lower RMS error when compared to vector fitting

Table 2. Comparison of Total Processing Times for Example 1 and Example 2 Using Different Blackbox Macromodeling Methods

| Macromodeling Method | Example 1 Processing Time [s] | Example 2 Processing Time [s] |
|---|---|---|
| AG | 2.3896 | 12.0952 |
| Vector Fitting (VF) | 67.0386 | 611.3338 |

As shown in Table 2, the AG model was able to process the data for both examples in approximately 2.4 and 12.1 seconds respectively, compared to the 1.1 minutes and 10.2 minutes required to run vector fitting. The vector fitting algorithm was set to use 20 iterations per model order. Conversely, AG did not require these SK-like iterations at each model order. The data was processed using MATLAB on a laptop with an Intel Core i7 CPU.

## V. Discussion

A complete convergence analysis is not yet available for SK iteration [4] or AAA [6]. The residual error does not monotonously decrease with the number of iterations or the number of poles. A theoretically better-studied method is the Levenberg-Marquardt approach that leads to the Whitfield estimator [14] for rational functions. Alternatively, fitting the derivative of the objective function in the instrumental variable approach aims to find the local minima [15]. These approaches have however not found widespread use, as they are either considered to be slow [16], [17] or not a significant improvement over SK iteration [18]. The convergence of AG especially in the presence of noise requires further research. Vector fitting as an example may fail to locate the poles for high noise levels [19] due to the presence of spurious poles. We plan to study the performance of AG in avoiding spurious poles for such data.

A commonly desired property for the rational function is that it should have real coefficients and stable poles. This is achieved in the current implementation by the underlying ORA algorithm. Passivity on the other hand is typically not preserved in rational approximation of passive microwave network parameters. It can be enforced if necessary as a post-processing step by perturbation of the residues, as commonly done in existing rational approximation approaches.

## VI. Conclusion

We presented a new adaptive generation (AG) approach to linearize the least squares problem and estimate the model order for rational function approximation. AG is an alternative to the celebrated Sanathanan-Koerner (SK) and adaptive Antoulas-Anderson (AAA) algorithms to compute rational function approximations. While further testing is needed, initial results are promising in generating fast and accurate rational approximations by the new AG approach.

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