Normal Incidence Scattering of Waveguide-Like Periodic Structures in Scalar 2D-FEM Mode-Matching Extracting the Frequency Dependence

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***Abstract*— The problem of scattering by periodic structures composed of waveguide-like unit cells can be tackled from a Mode-Matching perspective. Existing approaches for hybridisation through this technique using Floquet modes and the 2D Finite Element Method are based on full-wave representations of the field and do not consider any efficiency improvements in a wide-band frequency sweep, a typical case of application. In this work we present an alternative, holistic fast formulation directly derived from scalar potentials, which is also frequency independent up until the last step of the process of computing the generalised scattering matrix. It is formulated for incident angles normal to the periodicity plane, which are typically those first considered in the design workflow.**

***Keywords*— Frequency Selective Surfaces, Polarisation Selective Surfaces, Mode-Matching, Finite Element Method**

I. Introduction

Frequency Selective Structures (FSSs) and Polarisation Selective Structures (PSSs) [1] are one of the areas of research in microwave engineering currently registering a significant amount of activity. This is mainly due to their capability of modelling complex frequency responses from relatively simple geometries, which can be manufactured using novel additive techniques [2]. These structures are typically defined as periodic grids derived from a unit cell which is replicated along a plane, and simulated taking electromagnetic plane waves as the incident and transmitted fields [3].

On the one hand, the Floquet theorem aids in modelling, from a Computer Aided Design (CAD) perspective, spatially limitless plane waves considering a single unit cell from the infinite grid. On the other hand, waveguide theory can help in describing the electromagnetic field inside thick, waveguide-like cells composed of one of more sections, using modal analysis [4] and, for instance, a numerical technique such as the two-dimensional Finite Element Method (2D-FEM) to compute the modes [5]. This approach is one of the most preferred thanks to its robustness and ability to deal with arbitrary cross-sections. Other possible techniques for hybridisation are, for instance, analytic formulas for canonical waveguides and Boundary Integral-Resonant Mode Expansion (BI-RME) [6].

To integrate both of these tools, Mode-Matching [7], a technique for characterising discontinuities through a Generalised Scattering Matrix (GSM) [8], has been proposed in the past, obtaining very satisfactory results. This hybrid approach combines the strength of Floquet waves with said numerical methods, using a modal field expansion on both sides of the step and computing mode-by-mode couplings. In this regard, existing works provide a general way for obtaining modal cross-products in combination with FEM by using the transversal components of the fields [9], [10].

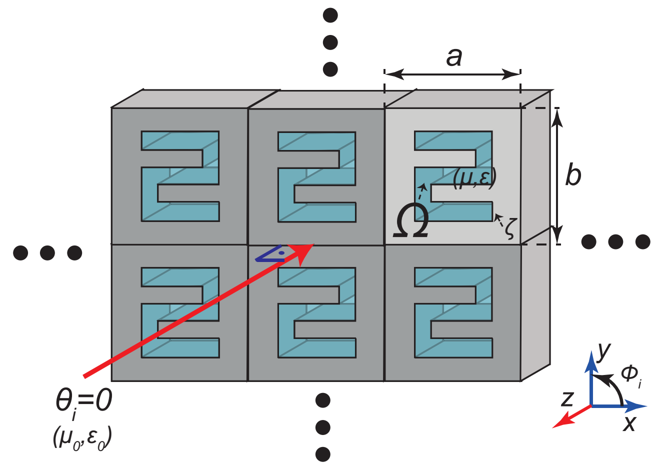

Fig. 1. A generic Frequency Selective Surface composed of waveguide-like (thick) unit cells laying on the xy plane filled by an homogeneous dielectric. The incidence angle is normal to the surface, i.e. is aligned with the z direction.

In this work we propose a convenient formulation for hybridizing Floquet waves with 2D-FEM through Mode-Matching. Intended for fast sweeps and to aid in the optimization of FSS and PSS, we take scalar Finite Elements instead of vectorial elements in the discontinuity plane as the starting point, considering the smallest possible representation of the problem, and isolate the frequency dependence so that, for incident angles equal to the normal direction, results are obtained fast and accurately. A handful of results are discussed, as well as the computational performance of the method.

II. Background

First, we will define the problem in both sides of the discontinuity, later to be integrated in a Mode-Matching scheme. The conventionalism given in [6] serves as a starting point, where Floquet waves are described using scalar potentials, in analogy to what was first proposed in waveguide theory [4].

A. Floquet modes

Appling the Floquet theorem to a unit cell yields generalised Transversal Electric (TE_{mn}) ($\psi = h_z$) and Transversal Magnetic (TM_{mn}) ($\psi = e_z$) modes, which are solution to the equation $\Delta_t\psi_{mn}+k_{mn}^2\psi = 0$ in the rectangular domain $a \times b$ with periodic boundary conditions imposed on the edges of the unit cell (see Fig. 1):

$$\psi_{mn}(x,y) = \frac{-j}{k_{mn}\sqrt{ab}} e^{-j(k_m x + k_n y)}. \quad (1)$$

where Δ_t is the transversal Laplacian operator. The parameters (see also Fig. 1) are

$$k_m = k_0 \sin\theta_i \cos\phi_i + \frac{2m\pi}{a}, \quad (2)$$

$$k_n = k_0 \sin\theta_i \sin\phi_i + \frac{2n\pi}{b}, \quad (3)$$

$$k_{mn} = \sqrt{k_m^2 + k_n^2}, \quad (4)$$

where $k_0 = \omega\sqrt{\mu_0\epsilon_0}$ is the vacuum wavenumber and $\{m, n\}$ are integers in the range $(-\infty, \infty)$, which must be truncated to a finite amount of terms for numerical purposes. It is assumed, for the sake of simplicity and without loss of generality for the remaining aspects of the formulation, that the angle between both periodicity directions is 90^o (and thus the unit cell is a rectangle of dimensions $a \times b$, using a coordinate system x, y at the rectangle corner). The complex z-wavenumber for each Floquet mode is computed as $k_z^{(f)} = \sqrt{k_0^2 - k_{mn}^2}$ for z-variation given by $e^{\pm jk_z z}$.

If there is normal incidence, i.e. $\theta_i = 0$ as in Fig, 1, then the two first propagating modes, TE_{00} and TM_{00}, can be considered as solutions of the equation $\Delta_t\psi_{00} = 0$, having the following potential functions:

$$\psi_{00}^{(TE)} = \frac{-x}{\sqrt{ab}}, \quad \psi_{00}^{(TM)} = \frac{-y}{\sqrt{ab}}. \quad (5)$$

Crucially, normal incidence also makes the frequency dependence vanish from k_m and k_n, and, as a result, from ψ_{mn} in (1). Hence, in this case modal functions may be considered only once for a wideband simulation.

B. *Waveguide modes*

Modes in homogeneous waveguides enclosed by perfect conductor can be obtained by solving the 2D Helmholtz equation $\Delta_t\psi + k_c^2\psi = 0$ in Ω as in Fig. 1 (again, $\psi = h_z$ for TE modes and $\psi = e_z$ for TM modes) with the following boundary condition on the surrounding conductor:

$$\nabla_t\psi_{TE} \cdot \hat{\boldsymbol{n}}|_\zeta = 0, \quad \psi_{TM}|_\zeta = 0. \quad (6)$$

The equation is to be solved using scalar 2D-FEM, where ψ is discretized and obtained as a weighted sum of basis functions l_i: $\psi = \sum_{i=1}^K v_i l_i$, where v_i are the Degrees of Freedom (DOFs). The final eigenvalue problem is $(\boldsymbol{S} - k_c^2\boldsymbol{T})\boldsymbol{v} = \boldsymbol{0}$, where matrices $\boldsymbol{S}$ and $\boldsymbol{T}$ have the following entries ij [5]:

$$S_{ij} = \iint_\Omega \nabla l_i \cdot \nabla l_j \, d\Omega, \quad T_{ij} = \iint_\Omega l_i l_j \, d\Omega. \quad (7)$$

Note that modal fields obtained through this approach do not have a frequency dependence in any case, as opposite to other formulations that compute the complex z-wavenumbers $k_z^{(w)} = \sqrt{k^2 - k_c^2}$ as the eigenvalues of the problem (taking harmonic propagation $e^{\pm jk_z z}$). $k = \omega\sqrt{\mu\epsilon}$ is the wavenumber of the homogeneous medium inside the waveguide.

III. Mode-Matching Formulation

A. *Projecting Floquet modes onto FEM basis functions*

To compute the modal cross-products in Mode-Matching, it is first necessary to project all the information to the same function space. We start from (1), for which we will obtain, for each of the Floquet modes, the values of their DOFs in the 2D-FEM scheme. For a generic mode of indices mn:

$$\sum_{i=1}^{K} u_i l_i = \frac{-j}{k_{mn}\sqrt{ab}} e^{-j(k_m x + k_n y)}. \quad (8)$$

Now we apply Galerkin, which yields the weak form of the previous equation

$$\iint_\Omega u_i l_i l_j \, d\Omega = \frac{-j}{k_{mn}\sqrt{ab}} \iint_\Omega e^{-j(k_m x + k_n y)} l_j \, d\Omega. \quad (9)$$

This represents a linear system of equations, which can be written as:

$$\boldsymbol{T}\boldsymbol{u}_{mn} = \boldsymbol{f}_{mn}, \quad (10)$$

where $\boldsymbol{T}$ is the same matrix as in (7) and $\boldsymbol{f}$ a column vector with index i obtained through

$$f_{mn,i} = \frac{-j}{k_{mn}\sqrt{ab}} \iint_\Omega e^{-j(k_m x + k_n y)} l_i \, d\Omega. \quad (11)$$

B. *Extracting the frequency dependence in Mode-Matching*

The step discontinuity problem at $z = 0$ in Fig. 1 can be solved using Mode-Matching [7], [8]. The fields are expanded at each side of the step with the waveguide (subscript w) and Floquet (subscript f) modes, respectively, using amplitudes $\boldsymbol{a}$ for the incident and $\boldsymbol{b}$ for the reflect waves. Then, the boundary conditions are applied by means of a Galerkin method, assuming $\iint_\Omega \nabla\psi_i \cdot \nabla\psi_i \, d\Omega = 1$ for the waveguide modes (note that (1) and (5) already satisfy this), leading to:

$$\begin{cases} (\boldsymbol{a}_f + \boldsymbol{b}_f) = \boldsymbol{X}_c\,(\boldsymbol{a}_w + \boldsymbol{b}_w) \\ \boldsymbol{X}\,(\boldsymbol{a}_f - \boldsymbol{b}_f) = (\boldsymbol{a}_w - \boldsymbol{b}_w) \end{cases}, \quad (12)$$

To minimise the amount of operations needed to compute each frequency point we separate the cross-product matrix $\boldsymbol{X}$ appearing in (12) into a frequency-independent normalized matrix $\bar{\boldsymbol{X}}$, so that $\boldsymbol{X} = \boldsymbol{Z}_w^{1/2}\bar{\boldsymbol{X}}\boldsymbol{Y}_f^{1/2}$ and $\boldsymbol{X}_c = (\boldsymbol{Z}_w^{1/2}\bar{\boldsymbol{X}}^*\boldsymbol{Y}_f^{1/2})^T$, (superscript T indicating transpose, and $*$, complex conjugate). $\boldsymbol{Z}$ and $\boldsymbol{Y} = \boldsymbol{Z}^{-1}$ are diagonal modal impedance and admittance matrices for the corresponding modes, both frequency dependent. All impedances might be obtained with the following formulae, where $\eta = \sqrt{\mu/\epsilon}$, and ξ identifies with both $k_z^{(f)}$ and $k_z^{(w)}$ [8]:

$$Z_{TE,i} = \frac{k\eta}{\xi_i}, \quad Z_{TM,i} = \frac{\xi_i\eta}{k}, \quad Z_{00} = \eta. \quad (13)$$

Z_{00} refers to the impedance of the TE_{00} and TM_{00} Floquet modes, which have a cutoff wavenumber $k_{00} = 0$.

Finally, the GSM from which transmission and reflection parameters for Floquet modes can be extracted is directly obtained the following way, where $\boldsymbol{F} = 2(\boldsymbol{I}_w + \boldsymbol{X}\boldsymbol{X}_c)^{-1}$ ($\boldsymbol{I}$ being the identity matrix of appropriate dimensions):

$$\boldsymbol{S}_{GSM} = \begin{bmatrix} \boldsymbol{X}_c\boldsymbol{F}\boldsymbol{X} - \boldsymbol{I}_f & \boldsymbol{X}_c\boldsymbol{F} \\ \boldsymbol{F}\boldsymbol{X} & \boldsymbol{F} - \boldsymbol{I}_w \end{bmatrix} \quad (14)$$

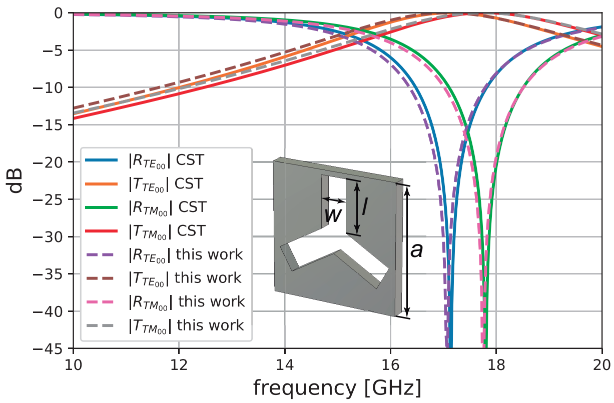

Fig. 2. Transmission and reflection parameters of the TE_{00} and TM_{00} modes for the FSS from [12], shown in the figure. The dimensions are $a = b = 10$ mm, $l = 4.04$ mm and $w = 2.0$ mm. The metal thickness is $t = 1.0$ mm.

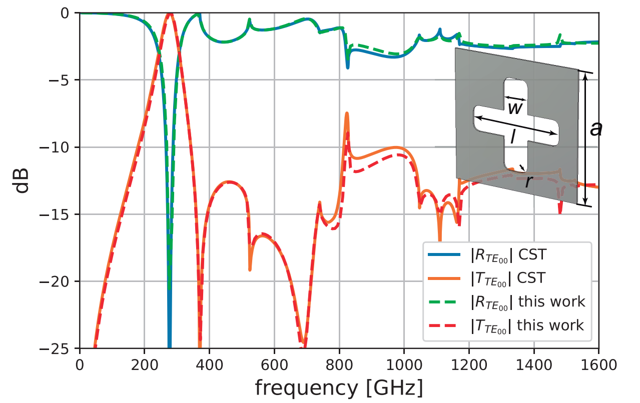

Fig. 3. Transmission and reflection parameters of the TE_{00} mode for the FSS shown in the figure, taken from [6]. The dimensions are $w = 160$ μm, $L = 570$ μm, $a = b = 810$ μm and $r = 40$ μm, and the metal thickness is $t = 10$ μm.

C. Normalised frequency-independent cross-product matrix for normal incidence

Lastly, thanks to using modal functions derived from scalar potentials, we now show how to compute the frequency-independent normalised cross-product matrix $\bar{\boldsymbol{X}}$. We can arrange Floquet modes depending on their nature, so that $\boldsymbol{u}_{00} = [\boldsymbol{u}_{00}^{(TE)}, \boldsymbol{u}_{00}^{(TM)}]$, which must be considered separately, $\boldsymbol{u}_{TE} = [\boldsymbol{u}_i^{(TE)}]_{i=1,\ldots,N_{TE}}$, and $\boldsymbol{u}_{TM} = [\boldsymbol{u}_i^{(TM)}]_{i=1,\ldots,N_{TM}}$, where N_{TE} and N_{TM} are the total number of Floquet TE and TM modes considered for the Mode-Matching. An analogous procedure can be done for waveguide eigenfunctions $\boldsymbol{v}_i$. Through this convention it is possible to compactly define $\bar{\boldsymbol{X}}$ as a block matrix:

$$\bar{\boldsymbol{X}} = \begin{bmatrix} \boldsymbol{v}_{TE}^T \boldsymbol{R} \boldsymbol{u}_{00} & \boldsymbol{v}_{TE}^T \boldsymbol{S} \boldsymbol{u}_{TE} & \boldsymbol{v}_{TE}^T \boldsymbol{R} \boldsymbol{u}_{TM} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{v}_{TM}^T \boldsymbol{S} \boldsymbol{u}_{TM} \end{bmatrix}, \tag{15}$$

where matrix $\boldsymbol{S}$ is the same as shown in (7) and $\boldsymbol{R}$ has the following entries ij [11]:

$$R_{ij} = \iint_{\Omega} \nabla_t l_i \times \nabla_t l_j \cdot \hat{\boldsymbol{z}}\, d\Omega. \tag{16}$$

It is important to stress that, using the proposed scheme, $\bar{\boldsymbol{X}}$ must be computed only once for a single cross-section, only having to recompute $\boldsymbol{X}$ and (14) for each frequency, substantially reducing the computational cost. The proposed holistic approach also allows for efficiently reusing the matrices from (7) in (10) and (15), thanks to considering every step of the problem as a part of its total.

IV. Results

To test the formulation, first we take a simple example from the literature, a classic unit cell from [12]. Its authors emphasize that the FSS is made of a thick metallic screen, i.e., the waveguide effect is not negligible. The parameters for the slots, shown as an inset in Fig. 2 are $a = b = 10$ mm, $l = 4.04$ mm and $w = 2.0$ mm. The thickness is $t = 1.0$ mm. In this figure, the reflection (R) an transmission (T) parameters for the first two propagating modes $R_{TE_{00}}$, $T_{TE_{00}}$, $R_{TM_{00}}$, $T_{TM_{00}}$ are compared with the presented method against CST Microwave Studio (CST MWS). Cross-coupling between these modes is not shown due to being negligible. A very good agreement is found between simulations.

A more challenging example from a simulation standpoint is the shape in Fig. 3 (inset). In this problem, which was taken from [6], the unit cell is a cross with rounded edges, which serves as a quasi-optical filter. A wide band analysis was desired, which, along with the high amount of Floquet modes required for simulation, represents a challenge for commercial 3D-FEM software. The dimensions of the slots are $w = 160$ μm, $L = 570$ μm, $a = b = 810$ μm and $r = 40$ μm. The thickness of the metal is $t = 10$ μm.

The results shown in Fig. 3 refer to the TE_{00} response in transmission and reflection, which are identical to the TM_{00} due to the symmetry of the problem. The formulation was also compared against CST MWS, obtaining very satisfactory results for the entire band. To perform a fair comparison, first-order symmetry properties of the modes were not considered as an optimisation for the Mode-Matching. Crucially, CST MWS took on average around 30 s to compute each frequency point considering 122 Floquet modes (up until modes $TE/TM_{\pm 3 \pm 3}$, which have a cutoff frequency of 1571 GHz), while the proposed method took only 0.01 s for each frequency point, considering 402 Floquet modes and 32 waveguide modes for the Mode-Matching procedure, thanks to the presented isolation of the frequency dependence.

V. Conclusion

Mode-Matching provides a robust and efficient way of analysing polarisation or frequency selective structures with waveguide-like unit cells. We have shown that it is possible, using scalar potentials and under normal incidence, to extract the frequency dependence from the formulation. This holistic approach greatly optimizes the entire process of simulating a wide-band response and can serve as a tool for the first stages of optimization of various designs, as shown in the examples.

Acknowledgment

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