WMG: Introduction/Survey of Nonlinear Dynamics Perspective for Microwave Engineering Applications

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**Presentation Topics**

- **The Dynamical System Perspective**
  - Classical Dynamical Systems
  - Stability & Bifurcation

- **Exploitation of Nonlinear Dynamics**
  - *Key Application Basis*: Chaos and Chaotic Synchronization
  - Chaos-Based RF Communications
  - Other Application Areas

- **Summary & Resources**
The Dynamical System Perspective
Linear vs. Nonlinear Paradigm I

- **Linear methodology provides first-order view of naturally nonlinear world (e.g., electrical circuits).**
  - Dominant engineering design approach (taught and practiced)
    - *Field very mature and settled*
    - *Problems tractable with general solutions (universality)*
  - *Circuit examples:* Linear resistors/capacitors/inductors, classical filters, low noise amplifiers, power amplifiers under small-signal excitation
  - Characterized by very simple qualitative behaviors stemming from fundamental
    **Principle of Superposition:** *Response to sum of stimuli is sum of responses to each stimulus*
Linear vs. Nonlinear Paradigm II

- **Nonlinear methodology provides detailed view by addressing higher-order effects that can no longer be ignored/approximated with linear approaches.**
  - Reality is modern circuit designs are replete with nonlinearities that must perform under ever more stringent conditions/performance demands
    - **Examples:** Diodes, transistors, oscillators, high power amplifiers, frequency converters, frequency multipliers/dividers, phase-locked loops, parametric devices
  - Universe of qualitative behaviors/effects fundamentally beyond linear case since principle of superposition does not hold, such as:
    - Simultaneous multiple steady states/operating points
    - Periodic behaviors or oscillations, synchronization and frequency entrainment
    - Qualitative behavior/stability changes with parameter variations (bifurcation)
      - Changes can also vary with parameter variation polarity (hysteresis, jumps)
    - Complex dynamical behavior that is quasi-periodic, chaotic, turbulent
  - New methodologies provide next evolutionary step in engineering analysis/design that either mitigate (current focus) or exploit nonlinear effects
    - Exploitation aspect leading to whole new nonlinear engineering discipline
Dynamical Systems Introduction

• A dynamical system is one in which a set of internal parameters (called states) obey a set of temporal rules (usually involving time derivatives).

• Study of dynamics has evolved into three disciplines:
  – Applied dynamics: Modeling of natural phenomena (Galileo, Kepler, Rayleigh)
  – Mathematical dynamics: Classical ordinary differential equations (ODEs) to geometry and topology (Newton, Poincaré, Thom)
  – Experimental dynamics: Laboratory measurements to computer simulations (Galileo, van der Pol, Lorenz)

• Fundamental modeling steps naturally give rise to dynamical system descriptions.
  – Basic assumption: Internal states described by a few observables
  – Mathematical idealization: Process leads to state-space model
  – Conventional interpretation: Match between actual and model states
State Variable Representations

- **Time series** — *Scalar variable versus time.*
  - Traditional engineering viewpoint, used especially in statistical contexts

- **Phase space** — *State variable with time as indirect parameter.*
  - Geometric perspective provides several benefits
Continuous (Analog) Dynamical Systems

- **State depends continuously on time** \( t \)
- **Governing rule is usually an ordinary or partial differential equation:**

\[
\frac{dx}{dt} \equiv \dot{x} = F(x) \quad (\text{autonomous or unforced ODE})
\]
\[
\dot{x} = F(x, t) \quad (\text{nonautonomous or forced ODE})
\]
\[
G(x, u_x, u_x^{(2)}, \ldots, u_x^{(n)}) = 0 \quad \text{PDE}
\]

- \( F \): vector field (smooth)
- \( x \): velocity
- \( F \) tangent to trajectory at \( x \)

Representative third-order ODE
Example: van der Pol Oscillator

- One of earliest chaotic circuit examples (B. van der Pol, 1927).
- Equivalent circuit:

\[
\begin{align*}
\dot{v} &= w - \Phi(v) \\
\dot{w} &= -\omega_0^2 v + \omega_1^2 B \sin\omega_1 t
\end{align*}
\]

\[v = \text{anode voltage, } \omega_0 = 1/\sqrt{LC} = \text{natural resonance}\]
\[\Phi(v) \equiv vR(v)/L, \quad R(v) = r + L\psi(v)/Cv\]
\[r = \text{ohmic resistance}\]
\[\psi(v) = "\text{oscillator characteristic}" \text{ of reactive triode}\]
Discrete (Digital) Dynamical Systems

- State depends on discrete sets of times $t_i$
- Governing rule is usually a difference equation (DE):
  \[ x_{n+1} = \Phi(x_n) \]  (autonomous)
  \[ x_{n+1} = \Phi(x_n, t_n) \]  (nonautonomous)

$\Phi$: state transition map
$x_n$: present state
$x_{n+1}$: next state

Representative third-order DE
**Example: Digital Phase-Locked Loop**

- **Basic model:**

  \[ s(t) = A \sin[\omega_1 t + \theta(t)] \]

  - For frequency-step input phase, first-order loop \([ D(z) = G_0 ]\) governed by autonomous, first-order DE:

    \[ \varphi_{k+1} = \varphi_k - A G_0 \omega_1 \sin \varphi_k + (\omega_1 - \omega_0) T \]  
    
    where

    \[ \varphi_k \equiv \theta_k - \hat{\theta}_k = \text{phase error} \in [0, 2\pi); \ T_k = T - y_{k-1} \]

    \[ \omega_0 = \text{nominal DCO frequency} \]

    \[ T \equiv 2\pi M/\omega_0 = \text{nominal sampling period} \]

- **Equation (#) is also called a sine-circle map — intricate dynamics, including chaos.**
Dynamical System Terminology I

- **Major classes of dynamical systems:**
  - **Forced** or **unforced**: Time-varying external stimulus present or not
  - **Dissipative** or **lossy**: Majority of physical systems (e.g., forced pendulum with friction)
  - **Conservative** or **lossless**: Quantum, classical Hamiltonian, idealized systems (e.g., planetary motion)
  - **Linear** or **nonlinear**: Governing vector field or state mapping obeys or does not obey the superposition principle:

\[
\mathbf{f}(\alpha \mathbf{v} + \beta \mathbf{w}) = \alpha \mathbf{f}(\mathbf{v}) + \beta \mathbf{f}(\mathbf{w})
\]
Dynamical System Terminology II

• **Other important concepts:**

  – *Orbit, trajectory:* Solution set of states for a dynamical system
  
  – *Flow:* Bundle of orbits often interpreted as fluid flowing in state space
  
  – *Transient* or *steady state:* Initial (short-term) or asymptotic (long-term) portion of temporal behavior
  
  – *Qualitative* or *quantitative:* Geometrical features (primary concern of dynamical system theory) or precise orbit locations
**Steady States I**

- **Constant orbit:** One in which the state variables remain fixed for all time.
  
  - **Equilibrium point:** Zero of vector field \( F(x_e, t) = 0 \) for all \( t \geq 0 \)
  
  - **Fixed point:** Invariant to iterated state transition map \( \Phi(x_e, k) = x_e \) for all \( k \geq 0 \)
    
    - Also can define \( m^{th} \)-order fixed point for \( m^{th} \)-iteration of \( \Phi \) \( \Phi^m(x_{em}, k) = x_{em} \) for all \( k \geq 0 \)
  
  - Stability of point determined by local orbit behavior

\[ C \text{ is an unstable saddle equilibrium point (2-D)} \]

*Illustrations for trajectories and sets are adapted from Abraham and Shaw, 1992.*
• **Periodic orbit**: One that repeats itself after some time $T$ (or some number of transitions $K$).

  – Also known as *closed orbit, cycle, or oscillation*
  
  – Stability also determined by behavior of nearby orbits

---

**Steady States II**

*Time series*

*Phase portrait*

**Stable periodic orbit (2-D)**
Poincaré Maps and Invariant Sets

- **First return or Poincaré maps** defined to effectively strobe a recurring flow in a continuous dynamical system.
  - Translates stability of closed orbit to \( m^{\text{th}} \)-order fixed point stability for map
  - Can also linearize flow along orbit to arrive at Floquet multipliers that determine stability
- **The inset (outset) of a steady state** is set of initial conditions such that their forward (reverse) flow approaches the steady state.
  - For a hyperbolic steady state (basically no neutral stability), inset (outset) becomes the stable (unstable) manifold for the steady state
Steady States III

• In 2-D phase spaces, only limited set of simple steady states formally possible (Peixoto, 1959).

• In higher dimensional phase spaces, whole zoo of steady-state behaviors can occur.
  – Experimentally observable intermittency, quasiperiodic, & chaotic behavior
    ▪ Directly corresponds to what are more commonly called spurious signals/parasitic oscillations
  – Unobservable homoclinic/heteroclinic connections involving unstable saddle-type sets that influence observable dynamics
    ▪ Strange attractor example: (third-order autonomous; Rössler, 1976)

\[
\begin{align*}
\dot{x} &= -(y + z), \\
\dot{y} &= x + \frac{1}{5} y, \\
\dot{z} &= \frac{1}{5} + z(x - \mu)
\end{align*}
\]

Initial state: \((-1,0,0)\)

\[\mu = 5.7\]
Stability Background & History

- **Next to circuit & system function, stability most important concern in engineering design.**
  - Focus is on DC operating points, transients, and steady-state behavior
  - Not only must stability be ensured, but with a robust margin as well
    - *Valid margin determination is key missing component of current stability analysis practices*

- **Dynamical systems provide natural context to address this concern.**
  - Stability concepts and qualitative analysis developed along with discipline
    - *Earliest principle dates back to Torricelli (1608–1647) concerning N-body stability*
  - Quantum advance came with Lyapunov (1892) for both stability definitions and analysis (largely unknown in West until ~ 1960!)
    - *More recent application to adaptive identification/control and feedback stabilization*
  - Alternative perspective provided by input-output stability theory pioneered more recently by Sandberg & Zames (1960’s)
    - *Conceptually clearer and easier to apply to distributed systems*
Basic Stability Concepts

• Three basic classes of stability have arisen in study of dynamical systems:
  – Lyapunov stability: Addresses local and asymptotic behavior of orbits about equilibrium points.
    ▪ Employs positive definite energy-like functions $V(x,t)$ with $dV(x,t)/dt$ of one sign and bounded (along all orbits) — hard part is to construct these
    ▪ Direct and indirect methods exist, latter involving linearization of nonlinear system
  – Input/Output stability: Treats dynamical system as input/output operator with possible dependence on initial state.
    ▪ Defines operator bounding and gain conditions involving its input and output
    ▪ Provides conditions to ensure concatenations of operators remain stable
    ▪ Primarily applicable to linear open loop & feedback systems (Nyquist-type criteria)
  – Structural stability: Addresses preservation of qualitative behavior given perturbations of governing vector field.
    ▪ Necessary condition is that all steady states are hyperbolic
    ▪ Structural instability provides seed for bifurcation
Lyapunov Stability & Linearization

- Provides fundamental basis for traditional notions of stability of equilibrium points.
  - Local forms are based on bounding initial conditions to give bound or limit to resulting orbit; global forms based on local forms
    - **Basic form:** Local stability in the sense of Lyapunov — Equilibrium point \( x_e = 0 \) is such that for all \( t_0 \geq 0 \) and \( \epsilon > 0 \), there exists \( \delta(t_0, \epsilon) > 0 \) such that
      \[
      \|x_0\| < \delta(t_0, \epsilon) \Rightarrow \|x(t, x_0)\| < \epsilon \quad \forall \ t \geq 0,
      \]
    where \( x(t, x_0) \) is the orbit starting at \( x_0 \) at \( t = t_0 \)
    - **Other forms:** uniform stability, asymptotic stability, uniform asymptotic stability, exponential stability, instability

- Fundamental result of Hartman-Grobman (1964) allows stability/local behavior of linearized dynamical system about equilibrium/fixed points to qualitatively reflect that of original nonlinear system.
  - **Basic condition:** Eigenvalues of vector field/state transition map Jacobian at point have nonzero real parts/non-unity modulus (i.e., they are **hyperbolic**)

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In general, a bifurcation occurs in a parameterized dynamical system if it becomes structurally unstable at a particular parameter value.

- **Local bifurcation** — Involves simple, distinct steady states (e.g., equilibrium/fixed points or periodic orbits) that are *non-hyperbolic* (i.e., cannot be linearized)

- **Global bifurcation** — Involves more complex steady states (e.g., homoclinic/heteroclinic orbits) that are structurally unstable and cause global phase portrait changes (non-local to steady state)
  - **Realm of catastrophe or singularity theory** pioneered by Thom (1973)

- Whole taxonomy developed for local bifurcations based on two fundamental results:
  - **Center Manifold Theorem (Kelley, 1967)** — Provides means of determining stability and associated center manifolds for non-hyperbolic equilibrium points
    - Series-based approximation of center manifold dynamical system provided later (Carr, Henry, 1981)
  - **Normal Form Method** — Provides systematic procedure to arrive at “simplest” description of given bifurcation type restricted to center manifold
Continuous Co-dimension One Bifurcations I

• **Four basic types of interest described locally by normal forms with one parameter (termed co-dimension one):**
  - *Saddle-node or fold* — Described by $\dot{x} = \mu - x^2$

  ![Saddle-node or fold diagram](image)

  **Note** — This and next two bifurcations:
  - Arise from simple real eigenvalue passing through zero
  - Concern DC-type equilibria

  ![Transcritical or exchange of stability diagram](image)

  - *Transcritical or exchange of stability* — Described by $\dot{x} = \mu x - x^2$
Continuous Co-dimension One Bifurcations II

- **Four basic bifurcation types (cont.)**
  - *Pitchfork* — Described by $\dot{x} = \mu x - x^3$

  ![Pitchfork Diagram]

  - *Hopf* — Described by $\dot{x} = -y + x(\mu - (x^2 + y^2))$, $\dot{y} = x + y(\mu - (x^2 + y^2))$

  ![Hopf Diagram]

  **Notes:**
  - Bifurcation here associated with pair of complex eigenvalues passing through $j\omega$-axis
  - Involves DC-type equilibria and periodic orbit
  - Underlying seed for oscillation creation
Other Bifurcations

• **Co-dimension one bifurcations also exist for discrete systems.**
  – In this case, one real or complex pair of characteristic eigenvalues pass through unit circle in complex plane, instead of jω-axis in continuous case
    ▪ **Unit circle and jω-axis fundamentally mark where system becomes non-linearizable**
  – Bifurcation types more numerous, including continuous system ones as subset
  – Real eigenvalue passing through −1 provides unique *period-doubling* or *subharmonic* bifurcation
  – Can also be used to characterize *periodic orbit bifurcations* (via Poincaré maps) that are much more complex in nature

• **Higher co-dimensions leads to exponential increase in bifurcation types and complexity.**
  – Complete cataloging only attempted for co-dimension 2 and 3
  – Unlikely in practical circuits/systems since more than one real or complex pair of eigenvalues (possibly degenerate) must become non-hyperbolic simultaneously

• **In addition, bifurcation routes often exhibit undesirable hysteretic/jump phenomena, motivating need for their detection.**
Example: Period Doubling in Rössler System

- **Period doubling is a common, fundamental route to chaos.**
  - Period-$2^n$ orbits appear as some *bifurcation parameter* varies — $n$ goes to infinity with finite value of this parameter
    - Actual time period of orbits goes up as $2^nT$, where $T$ is basic period
  - In Rössler system, occurs as $\mu$ increased ($\mu_\infty \approx 4.2$ where chaos first occurs):

```
\mu = 2.6  \quad 3.5  \quad 4.1  \quad 4.23  \quad 4.30  \quad 4.60
```

---

- Period-1
- Period-2
- Period-4
- Chaos about period-4
- Chaos about period-2
- Chaos about period-1
Despite fundamental dynamical system basis of stability, mainstream practical stability analysis essentially ignores this perspective (hence one reason for this workshop).

- In general, current stability analysis targets primarily linear systems, amplifiers, and oscillators
- In many cases, stability is determined simply by running a simulation and checking circuit power dissipations

Current practice primarily uses linear approaches applied to linear (or linearized) systems, often without regard to applicability/assumptions — 2 prime examples:

- Linville or Rollett stability criteria — Placed on scattering parameters for a 2-port to determine unconditional and conditional (potentially unstable) stability
  - Fundamentally a linear, sinusoidal steady-state approach that is often misapplied
  - Assumes (often not checked) unloaded N-node linear network for which 2-port is embedded does not contain poles in RHP

- Kurokawa oscillation/stability criteria — Provides impedance-based approach of ensuring oscillatory behavior and its stability
  - Issue is balance of positive and negative resistance can readily result in much more than simple oscillation for a nonlinear system (e.g., quasiperiodic or chaotic behavior)
  - Fundamental creation of oscillation through Hopf bifurcation completely missed here
Exploitation of Nonlinear Dynamics
What is Chaos?

- Chaos is bounded, random-like behavior in a deterministic dynamical system — that is, “noise” with an underlying order.

- Example — Progression from order to disorder in a flowing stream:

  Smooth laminar flow  Stable vortex detachment
  
  A               B

  Vortex detachment  Fully engaged turbulence
  
  C               D
Continuous Example: Lorenz Attractor

- One of the first strange attractors discovered in natural sciences (1963).
  - Third-order autonomous dynamical system modeling thermal convection/flow in viscous fluid or atmosphere ($\sigma$, $B$, $R$ are physical parameters):

  \[
  \begin{align*}
  \dot{x} &= \sigma(y - x), & \dot{y} &= Rx - y - xz, & \dot{z} &= -Bz + xy
  \end{align*}
  \]

Lorenz Attractor
($\sigma = 10$, $B = 8/3$, $R = 28$)

Sensitivity to Initial Conditions

\[
\begin{align*}
  y(0) &= 0 \\
  y(0) &= 0.01
\end{align*}
\]

\[(x(0), z(0)) = (10, 30)\]
Discrete Example: Logistic Map

- Originated as a population dynamics model (Verhulst, 1844 & 1847).
- Dynamical system (1-D map): $x_{n+1} = \mu x_n (1 - x_n) \equiv f(x_n)$, $0 \leq \mu \leq 4$
- Sample orbits from this map’s rich set of dynamics:

$$
\begin{align*}
\text{ORDER} & \\
& \quad x_{n+1} = f(x_n) \\
\text{CHAOS} & \\
& \quad x_{n+1} = f(x_n)
\end{align*}
$$

Stable fixed point $x_e$

$$
|f'(x_e)| < 1, \; \mu < 4
$$

Unstable fixed point $x_e$

$$
|f'(x_e)| > 1, \; \mu = 4
$$
Classical and Chaotic Synchronization

- **Classical synchronization (or entrainment) involves periodic orbits.**
  - **Driven or injection type** — Small input synchronization signal causes large signal with same frequency (harmonic or subharmonic): one-way information flow
  - **Coupled type** — Linked oscillators lock up with appropriate coupling strength: two-way information flow

- **Chaotic synchronization can also occur in both driven and coupled configurations.**
  - First discovered in driven type by Pecora & Carroll at NRL (1990)
  - **Basic idea** — Replicate master chaotic system (subsystem), or synthesize its inverse remotely, and link appropriately
  - Makes possible generalization of classical sinusoid-based communication systems
Chaotic Synchronization Methods

• **Method 1 — Unforced system unidirectionally driving stable subsystem**
  – Fundamental, first, and most mature approach
  – Initiated new field of research on chaos-based communications

• **Method 2 — Forced system unidirectionally driving identical forced system**
  – Later basic discovery also useful for communications

• **Method 3 — Forced system unidirectionally driving inverse system**
  – Synchronization here means reproduction of original forcing signal

• **Method 4 — Adaptive control systems**
  – Involves control chaos, allowing for a wide variety of receiver configurations

• **Method 5 — Couple identical chaotic systems with two-way link**
  – Simple generalization of classical form useful for network communications
Method 1: Master/Slave

- **Basic configuration** — One chaotic system (master) driving replicated subsystem (slave) through communications channel.

![Diagram of master/slave system]

- **Drive System (chaotic)**
  - Driving variables
  - \( \mathbf{v} \) subsystem
  - \( \mathbf{w} \) subsystem

- **Response System**
  - \( \mathbf{w}' \)-system

- **Communications channel**

- **Decomposition**
  \[
  \dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})
  
  [\mathbf{x} = (\mathbf{v}, \mathbf{w})]
  
  \dot{\mathbf{v}} = \mathbf{g}(\mathbf{v}, \mathbf{w})
  
  \dot{\mathbf{w}} = \mathbf{h}(\mathbf{v}, \mathbf{w})
  
  \mathbf{w}' = \mathbf{h}(\mathbf{v}, \mathbf{w}')
  
  \]

- **First-order synchronization equation**
  \[
  \frac{d(\Delta \mathbf{w})}{dt} = D\mathbf{h}(\mathbf{v}, \mathbf{w}')\Delta \mathbf{w}
  
  \Delta \mathbf{w} = \mathbf{w}' - \mathbf{w}
  
  \Delta \mathbf{w} \to 0 \text{ with } t \to \text{ synchronization}
  
  ➢ **Response system’s parameters allow for real-world tolerances**
  
  ➢ **Precise conditions exist to determine appropriate stable subsystem needed for successful synchronization**
  
  ➢ **Several different configurations possible when such systems are cascaded**
Example of Method 1

- **Lorenz’s system ― Prototypical chaotic system to model convection.**

  Drive System
  \[ \dot{x} = \sigma(y - x) \]
  \[ \dot{y} = -xz + Rx - y \]
  \[ \dot{z} = xy - Bz \]

  Chaotic regime: \( \sigma = 16 \), \( B = 4 \), \( R = 45.92 \)

  Response System
  \[ \dot{y}' = -xz' + Rx - y' \]
  \[ \dot{z}' = xy' - Bz' \]

  \( x \)-driven

  ![Graph showing z(t), z'(t) over time](image-url)
Chaos-Based Modulations

- Chaotic synchronization has given rise to several modulations that can provide communications privacy/security capabilities:
  - **Additive chaotic masking**: Earliest form done with cascaded arrangement
    - Also recently demonstrated in optical realm by EU OCCULT project (Larger, 2004)
    - Caution: In principle, easily unmasked with nonlinear dynamics techniques
  - **Chaotic switching**: Can be thought of as attractor-shift keying
  - **Chaotic source modulation**: Essentially chaotic analog modulation of sinusoidal source using nonautonomous or inverse synchronization
    - Used in Aerospace chaos-based communications development effort
  - **Multiplicative chaotic mixing**: Analog of traditional spread spectrum
  - **Parametric modulation**: Indirect coding with chaotic multiplexing capability
  - **Forcing modulation**: Message inserted as independent source in transmitter
  - **Generalized modulation**: Message/chaos combined in general invertible manner
Example of Chaotic Masking Modulation

- Cascaded configuration with high immunity to noise interference in channel based on Lorenz’s system (Oppenheim et al., 1992).
  - Found later to be easily unmasked with nonlinear dynamics techniques
Chaos-Based Cryptography

• **Motivated by fact that diffusion and confusion inherently provided by chaotic dynamics.**

• **Fundamentally different from conventional methods:**
  – Conventional cryptography requires quantized message samples taken at discrete times
  – Chaotic cryptography can operate on continuous message samples from discrete times or on the entire message waveform

• **Two basic approaches to chaotic cryptography have been proposed:**
  – *Data encryption* — Uses invertible chaotic maps on 1-D and 2-D data — message “disintegrates” in a recoverable manner
  – *Modulation encryption* — Chaotic modulation used to “hide” and transmit message (analog or digital, encrypted or unencrypted) in a recoverable manner

• **Specialized cryptanalysis methods also developed for these approaches.**
  – Implies tools already in place to evaluate security of any proposed scheme
  – Tools may also uncover vulnerabilities in conventional cryptography
**Image Encryption Example**

- **3-D Baker map** with simple grey level permutation leads to uniform image histograms despite non-uniform original image histogram (Fridrich, 1998).

*Original Test Image (472 x 472 pixels, 256 grey levels)*

*One Iteration*

*Nine Iterations*
• **Unique nonlinear effect where added noise can be “tuned” to enhance a desired signal property (e.g., SNR).**
  
  – First proposed as explanation for Earth’s ice ages (Benzi, 1981)
  
  – Turning-point experimental demo for bistable laser (McNamara, 1988) started serious theoretical and experimental study of SR
  
  – Occurs in continuous and discrete systems (physical, chemical, and biological)
  
  – System typically exhibits a threshold effect (e.g., bi-stability, multi-stability, or chaos-chaos intermittency) induced by adding noise to forcing signal

* Mitiam & Kosko, 1998
**Stochastic Resonance Applications**

- **SR opens up new area of noise engineering design.**
  - Noise added to improve human perceptions (audio, visual, tactile)
  - Noise shaping to fit signal environment or vice versa
  - Signals used moving beyond periodic to aperiodic and modulated waveforms
  - Noise type used similarly moving beyond Gaussian
  - Effect also found to be enhanced using arrays of resonators
  - Results demonstrated for chaotic circuits indicate disruptive potential for future communications applications
  - Recent swell of activity in using SR for receiver detection enhancement

*Comte & Morfu, 2003*
• Beyond RF communications, chaos-based communications based on semiconductor lasers have also been developed and demonstrated.
  – Motivations similar to RF case: bandwidth efficiency, multi-user capabilities, natural large-signal operation, privacy/security
  – Chaotic behavior easily generated through optical injection, optoelectronic feedback, or in an optical cavity
  – Synchronization and modulation methods are more limited in variety
  – Sample codec based on Mach-Zehnder modulators (Larger, 2004):
Chaotic Radar

• Application motivated by natural LPI property — target scanned with low-level “noise” yet range and direction can be obtained.

• No worse than conventional radar signals when processed conventionally, but can be better under several scenarios:
  – Ideally suited for continuous transmission (versus traditional pulsed operation)
  – Aperiodic and capable of information encoding, signals are separable
  – Wide bandwidth naturally gives enhanced range resolution
    ▪ Can be designed to further enhance both range (by lowering sidelobes) and rate resolution — provides ambiguity function near ideal, but impractical white noise case (Carroll, 2005):
Cellular Nonlinear Networks (CNNs)

- Consists of 2-D (or higher-dimensional) array of nonlinear dynamical systems (cells) noninearly locally coupled (cloning template).
  - Always discretized in space, time evolution discrete or continuous
  - Each cell has input, output, and state

- Ushered in new paradigm in massively parallel computing.
  - Programmable as analog computer (analogic) with computational power reaching Tera OPS equivalent digital processing
  - Readily implemented in VLSI chip form using automated design tools

- Applications range from biological modeling (e.g., bionic eye) to image processing where each cell (or layers thereof) is associated with each pixel.
  - Can perform feature extraction, motion detection, path tracking, resolution enhancement, de-blurring, compression, etc.
  - Potentially disruptive impact on image/video processing for satellite/UAV applications

1 Cruz and Chua, 1991
2 Chua and Roska, 1993
Other Applied Nonlinearity

• Several other nonlinear technique areas also providing important tools for modeling, analyzing, and improving modern information systems.
  
  – **Fractals**: Provides powerful new framework capturing geometrical intricacies of natural phenomena.
    
    ▪ Involves important elements of self-similarity, repetitive iteration, and fractional dimension
    
    ▪ Fruitfully applied and even commercialized in application areas such as data/video compression, special media effects, frequency-independent antennas/arrays, random process and probability modeling, image halftoning, cluster and crack propagation analysis
  
  – **Solitons**: Localized (temporally or spatially) particle-like waves existing in nonlinear media and distributed circuits.
    
    ▪ Fiber optical solitons vigorously investigated for next generation long-haul optical communication systems

  – **Nonlinear system modeling**: Formal identification techniques needed to meet modeling fidelity/compensation effectiveness demands of future broadband commercial and military communication systems.
    
    ▪ High throughput, power and bandwidth efficiencies are basic drivers

• Applied nonlinearity under vigorous study in medical field for modeling, analysis, and control of biological systems (typically highly nonlinear).
Summary & Resources
Summary — Dynamical System Perspective

- **Field of dynamical systems provides fundamental framework for analysis/design of electrical & electronic systems.**
  - Unique perspective, plus vast accompanying set of qualitative/quantitative analysis techniques, provides powerful new tools for practicing engineers
  - Short-term application motivated by increased need to understand/mitigate nonlinear effects in current systems
    - *Prime/relevant example:* Provides for fundamental understanding and formal analysis of behavioral stability in microwave circuits and systems
  - Long-term application interest fueled by turning-point discoveries that provide for novel exploitation of nonlinear effects
  - Both application horizons enabled by constantly escalating computational capabilities
Nonlinear Discipline Lecture Series Link

• Recent groundbreaking general online lecture series, “From Memristors and Cellular Nonlinear Networks to the Edge of Chaos,” by Prof. Leon O. Chua (U.C. Berkeley) carried out covering:
  – Memristors — Missing **fourth** fundamental circuit element predicted in 1971 by Prof. Chua, realized in 2008 at HP Labs, and currently disrupting non-Von Neumann computing technologies
  – Cellular Nonlinear Networks — Fundamentally new analog computational paradigm co-invented by Prof. Chua
  – Local Activity and “Edge of Chaos” Principles — Essentially constituting a new **fourth** fundamental law of thermodynamics responsible for nature’s complexity
  – All topics heavily tied in with neural life sciences and fundamental physics

• **Replay link on YouTube originally presented at HP Labs:** [Chua's HP Labs Lectures on YouTube](https://www.youtube.com)
  – Also recently published in more complete book form by *World Scientific* (Singapore, 2021)
**Selected References I**

- **Dynamical Systems**
  


- **Stability & Bifurcations**
  

Selected References II

• Stability and Bifurcations (Cont.)
  


• Microwave Circuits & Systems Treatments
  


• Microwave Circuits & Systems Treatments (Cont.)

M. Odyniec (Ed.), *RF and Microwave Oscillator Design*. Boston: Artech House, 2002. [Provides good coverage of more advanced techniques; especially see Chapter 5 from V. Rizzoli’s group on HBM-based CAD implementations with optimization capability]

B. Parzen and A. Ballato, *Design of Crystal and Other Harmonic Oscillators*. New York: John Wiley & Sons, 1983. [Chapter 3 covers piezoelectric resonators (crystal-based oscillators)]


Selected References IV

- **Microwave Circuits & Systems Treatments (Cont.)**


Selected References V

• **Microwave Circuits & Systems Treatments (Cont.)**


• **Stability Analysis of High Efficiency Oscillators**

• **Stability Analysis and Stabilization of Power Amplifiers**


• **Class E/F Power Amplifier Design**


The scientist does not study nature because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful. If nature were not beautiful, it would not be worth knowing, and if nature were not worth knowing, life would not be worth living.

Henri Poincaré*

*Pioneering giant in nonlinear qualitative dynamics